(uniform surface temperature T_1) and hemisphere 2 (uniform surface temperature T_2) is expressed as

$$Q = A_1 \mathscr{F}_{12}\, \sigma(T_1^4 - T_2^4). \tag{4}$$

The overall exchanging factor $\mathscr{F}_{12}$ is [1]

$$\frac{1}{\mathscr{F}_{12}} = 2\left(\frac{1}{p} - 1\right) + \frac{1}{\bar{F}_{12}} \tag{5}$$

where p is the emissivity of the surface of hemispheres.

In terms of radiant heat-transfer coefficient h_r, equation (4) is approximately expressed as

$$Q = A_1 h_r (T_1 - T_2) \tag{6}$$

$$h_r = 4\sigma \mathscr{F}_{12} \bar{T}^3 \tag{7}$$

where $\bar{T} = (T_1 + T_2)/2$ in °K.

Substituting the Stefan–Boltzmann constant $\sigma = 4{\cdot}88 \times 10^{-8}$ (kcal/m^2 h °K^4) and equation (5), equation (7) is rewritten as

$$h_r = \frac{0{\cdot}1952}{2/p - 0{\cdot}264}\left(\frac{\bar{T}}{100}\right)^3 \text{kcal/m}^2\text{h°C}. \tag{8}$$

Most of the expressions for radiation heat-transfer coefficient have been defined on the basis of cross sectional area A_C.

$$\left.\begin{aligned} Q &= A_C h_r'(T_1 - T_2) \\ h_r' &= 4\sigma\psi \bar{T}^3 \end{aligned}\right\} \tag{9}$$

Thus, the formulae for ψ are compared as follows:

Investigator	Formula for ψ	
Damköhler [2]	$\dfrac{1}{(2/p) - 1}$	also used in literature [3, 4]
Schotte [5]	p	
Chen and Churchill [6]	$\dfrac{2}{a + 2b}$	a and b are experimental coefficients
This work	$\dfrac{2}{(2/p) - 0{\cdot}264}$	2 in numerator is the ratio A_1/A_C

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LAMINAR COUETTE FLOW WITH HEAT TRANSFER NEAR THE THERMODYNAMIC CRITICAL POINT

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NOMENCLATURE

- Br, Brinkmann number, $\dfrac{\mu_w U_0^2}{k_w(T_0 - T_w)}$;
- C_f, skin fraction coefficient, $2\tau_w/\rho_w U_0^2$;
- h, distance between upper and lower plates;
- k, thermal conductivity;
- Nu, Nusselt number, $\dfrac{q_w h}{k_w(T_w - T_0)} = -q_w^*$;
- P, pressure;
- q, heat flux;
- Re, Reynolds number, $\dfrac{\rho_w U_0 h}{\mu_w}$;

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T, temperature;
U_0, velocity of upper plate;
u, velocity;
y, normal coordinate;
μ, viscosity;
ρ, density;
τ, shearing stress.

Subscripts
c, conditions at critical point;
o, conditions at upper plate;
w, conditions at lower plate.

Superscript
$*$ parameter is nondimensional according to pertinent definition in text.

INTRODUCTION

THE MAGNITUDE of the wall heat transfer and skin friction in the flow of a viscous heat conducting fluid are known to be very dependent on the physical (thermodynamic and transport) properties of the fluid Thus, near the critical point of the fluid where the physical properties vary strongly with both pressure and temperature, one would expect that the skin friction and heat transfer would be extremely sensitive to the thermodynamic state of the fluid.

The present work examines the effects of the severe fluid transport property variations near the critical point on laminar Couette flow with heat transfer. The constant property solution for Couette flow is well known. Because of its simplicity and similarity with boundary layer type flow, Couette flow has been used to study various effects in viscous heat-conducting fluids. For example, the effects of dissociation and ionization have been studied by Liepmann and Blevis [1], the effects of binary mass addition by Knuth [2], and the combined effects of radiation and conduction by Grief [3] all using the Couette flow model. In addition, since only transport properties are involved, this model can be used to separate the influence of transport from thermodynamic properties.

ANALYSIS

Consider the laminar, steady, two-dimensional motion of a viscous heat-conducting fluid bounded by two infinite plane walls parallel to the x–z plane. The lower wall, located at $y = 0$, is stationary while the upper wall, located at $y = h$, is in uniform rectilinear motion parallel to the x axis with a velocity U_0 The lower wall is at a uniform temperature T_w, while the upper wall is at a uniform temperature T_0. The pressure is assumed to be constant and the normal or y component of velocity is assumed to be zero. It is further assumed that the fluid dynamic and thermodynamic properties of the flow do not vary with x. Under theses assumptions, the momentum and energy equations in nondimensional form become.

$$\frac{d\tau^*}{dy^*} = 0 \tag{1}$$

$$\frac{d}{dy^*}[Br(u^*\tau^*) - q^*] = 0 \tag{2}$$

$$\tau^* = \mu^* \frac{du^*}{dy^*} \tag{3}$$

$$q^* = -k^* \frac{dT^*}{dy^*} \tag{4}$$

where the nondimensional variables are

$$u^* = \frac{u}{U_0},\quad T^* = \frac{T - T_w}{T_0 - T_w},\quad y^* = \frac{y}{h},\quad k^* = \frac{k}{k_w},\quad u^* = \frac{\mu}{\mu_w},$$
$$\tau^* = \frac{\tau h}{\mu_w U_0},\quad q^* = \frac{qh}{k_w(T_0 - T_w)},\quad Br = \frac{\mu_w U_0^2}{k_w(T_0 - T_w)}.$$

The Brinkmann number (Br) is a measure of the ratio of viscous dissipation to heat transfer by conduction. For most fluids near the thermodynamic critical point, in practical situations where laminar flow will exist, the Brinkman number is of the order of 10^{-2} or lower. Although it could be neglected in the present analysis, the viscous dissipation term will be retained in the energy equation since it could be necessary in some other variable property situation and it does not greatly increase the complexity of the solution. The limiting case of zero Brinkman number will, however, be indicated later.

The boundary conditions are

$$\text{at } y^* = 0:\quad u^* = 0,\quad T^* = 0 \tag{5}$$

$$\text{at } y^* = 1:\quad u^* = 1,\quad T^* = 1\cdot \tag{6}$$

Directly integrating equations (1) and (2) and using the results in the subsequent integration of equations (3) and (4) produces the final results

$$\frac{q_w^*}{\tau_w^*} = -\left[\frac{Br}{2} + \int_0^1 \frac{k^*}{\mu^*} dT^*\right] \tag{7}$$

$$\tau_w^* = \int_0^1 \frac{k^* \, dT^*}{\sqrt{\left\{\left[\frac{Br}{2} + \int_0^1 \frac{k^*}{\mu^*} dT^*\right]^2 - 2Br \int_0^{T^*} \frac{k^*}{\mu^*} dT^*\right\}}} \tag{8}$$

$$u^* = \frac{1}{Br}\left[\left[\frac{Br}{2} + \int_0^1 \frac{k^*}{\mu^*} dT^*\right] - \sqrt{\left\{\left[\frac{Br}{2} + \int_0^1 \frac{k^*}{\mu^*} dT^*\right]^2 - 2Br \int_0^{T^*} \frac{k^*}{\mu^*} dT^*\right\}}\right] \tag{9}$$

$$y^* = \frac{\displaystyle\int_0^{T^*} \frac{k^*\,dT^*}{\sqrt{\left[\frac{Br}{2} + \int_0^1 \frac{k^*}{\mu^*} dT^*\right]^2 - 2Br \int_0^{T^*} \frac{k^*}{\mu^*} dT^*\Big\}}}}{\displaystyle\int_0^1 \frac{k^*\,dT^*}{\sqrt{\Big\{\left[\frac{Br}{2} + \int_9^1 \frac{k^*}{\mu^*} dT^*\right.^2 - 2Br \int_0^{T^*} \frac{k^*}{\mu^*} dT^*\Big\}}}}. \tag{10}$$

Equations (7) and (8) determine the heat transfer and shearing stress at the wall and equations (9) and (10) are sufficient to determine the velocity and temperature profiles. In the limit of $Br \to 0$ equations (7–10) reduced respectively to

$$\frac{q_w^*}{\tau_w^*} = -\int_0^1 \frac{k^*}{\mu^*} dT^*, \quad \tau_w^* = \frac{\int_0^1 k^*\,dT^*}{\int_0^1 \frac{k^*}{\mu^*} dT^*},$$

$$u^* = \frac{\int_0^{T^*} \frac{k^*}{\mu^*} dT^*}{\int_0^1 \frac{k^*}{\mu^*} dT^*}, \quad y^* = \frac{\int_0^{T^*} k^*\,dT^*}{\int_0^1 k^*\,dT^*},$$

In the general case (as well as in the limit $Br \to 0$), the solution is obtained in terms of quadratures involving the transport properties k and μ.

The constant property heat-transfer and skin friction expressions are

$$Nu = 1 + \frac{Br}{2}, \quad \frac{C_f Re}{2} = 1.$$

The constant property relation between heat transfer and skin friction is

$$Nu = \frac{C_f Re}{2}\left[1 + \frac{Br}{2}\right].$$

Velocity and temperature profiles have been obtained by numerical evaluations of the integrals involved in equations (9) and (10) for *para*-hydrogen near its critical point. The properties of *para*-hydrogen in current use in the vicinity of thermodynamic critical point were available to the authors as a standard computer subroutine STATE [4]. The transport properties for *para*-hydrogen are shown in Fig. 1 for a pressure ratio $P/P_c = 1.03$. The drastic changes in the properties with temperature near the critical point are obvious.

The resulting velocity and temperature profiles are shown in Figs. 2 and 3 for the pressure ratio $P/P_c = 1.03$ and for several values of the wall-to critical-temperature ratio. Similar curves have been obtained at higher pressure ratios

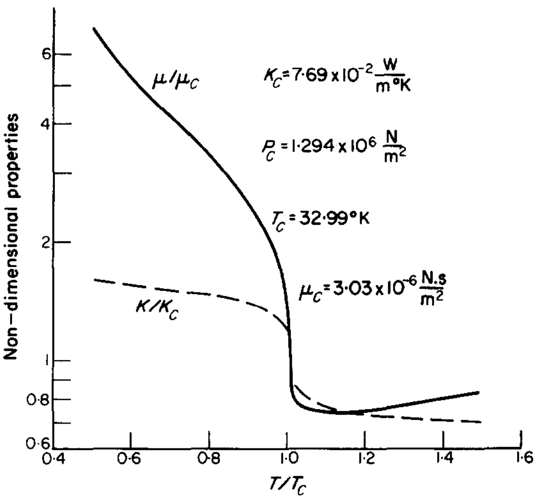

FIG. 1. Transport properties of *para*-hydrogen· $P/P_c = 1.03$; [4]·

and, as expected, the variations in the curves diminish as the pressure becomes further from the critical pressure. Since the shear is constant between the plates, it is not surprising that near the cooler wall ($y^* = 0$) where the viscosity is relatively high (see Fig. 1), the velocity gradient du^*/dy^* is relatively low. Near the $y^* = 1$ wall, where the fluid is relatively warm (and the viscosity is correspondingly lower); the velocity gradient is relatively high. The qualitative shape of the temperature profile can be explained by a similar consideration of the variation of thermal conductivity with temperature.

More interesting, however, are the skin friction and heat-transfer results shown in Fig. 4. Here the Nusselt number and the product $(C_f Re/2)_w$ are presented as functions of the dimensionless temperature $T_c^* = (T_c - T_w)/(T_0 - T_w)$, with pressure level as a parameter. It is convenient to present the results in terms of this dimensionless temperature since it is indicative of the temperature of the fluid between the walls relative to the critical temperature.

At large negative values of T_c^* (gas region), both the Nusselt number and the skin friction coefficient are close to the constant property values, which indicates that the rather weak temperature dependence of the properties in this region is adequately approximated by a constant property analysis. Since Nu_w and $(C_f Re/2)_w$, as defined, are for the stationary wall, one observes that as $T_c^* \to 0$ (i.e. $T_w \to T_c$) the Nusselt number and skin friction coefficient for the stationary wall both drop off sharply. At a pressure 3 per cent grater than the critical pressure, the Nusselt number is reduced by 34 per cent and the skin friction by 44 per cent at $T_c^* = 0$. Interestingly, the corresponding change in thermal conductivity is 200 per cent, and in viscosity is of the order of 400 per cent. Finally, as T^* increases away from zero, the Nusselt number returns to constant property, which is

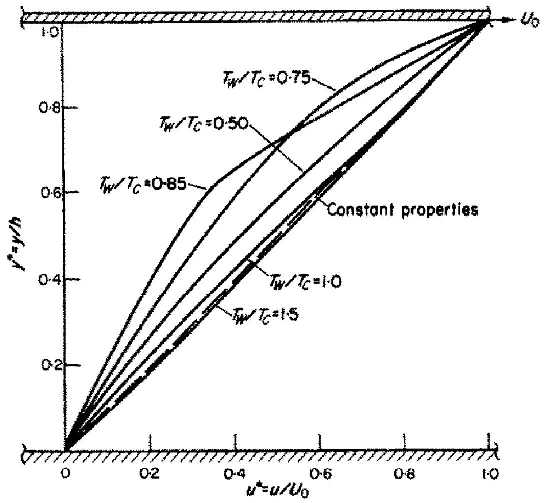

FIG. 2. Velocity profiles for *para*-hydrogen. ($Br = 0.01$; $P/P_c = 1.03$; $\Delta T/T_c = 0.30$).

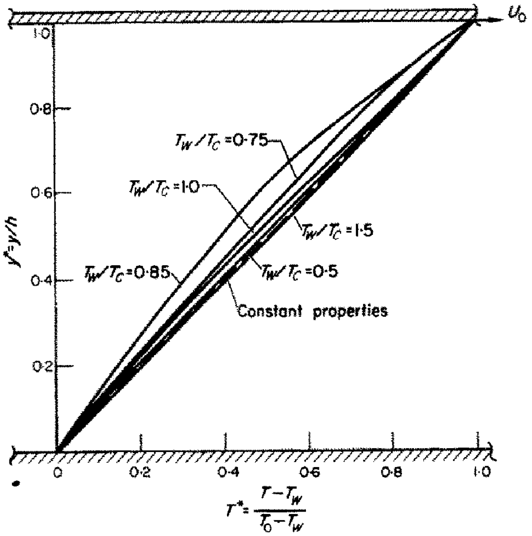

FIG. 3. Temperature profiles for *para*-hydrogen. ($Br = 0.01$; $P/P_c = 1.03$; $\Delta T/T_c = 0.30$).

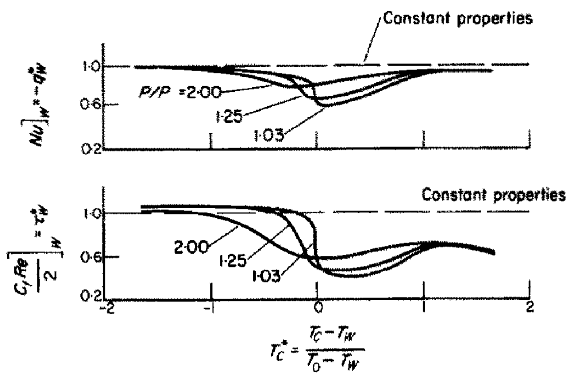

FIG. 4. Heat transfer and skin friction for *para*-hydrogen. ($Br = 0.01$; $\Delta T/T_c = 0.30$).

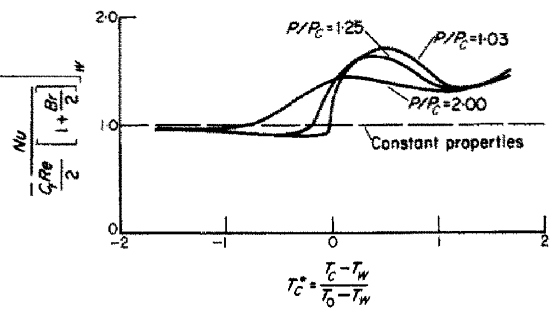

FIG. 5. Heat transfer and skin friction analogy for *para*-hydrogen. ($Br = 0.01$; $\Delta T/T_c = 0.30$).

consistent with the behaviour of the liquid thermal conductivity (Fig. 1). The skin friction coefficient, however, does not come back to the constant property value, which reflects the rather strong variation of liquid viscosity with temperature.

In Fig. 5, the constant property analogy between heat transfer and skin friction is seen to hold as it is brought close to the critical point. If, however, part of the fluid between the plates is at the critical temperature ($0 < T_c^* < 1$), the deviation from the analogy is significant. The trend in the liquid region is not clear, but $T_c^* = 1.6$ is very near the freezing point for hydrogen and the curves could not be extended.

CONCLUDING REMARKS

In laminar flow, the Couette flow model examines only the influence of the transport properties μ and k on heat transfer and skin friction; an analysis of this model near the thermodynamic critical point produces the following results.

1. Although the influence of the variation of transport properties near the critical point is to sharply decrease heat transfer, this decrease is nowhere near the magnitude of the property variations themselves

2. The low Brinkman number (10^{-2} or less) associated with the critical point indicates that viscous dissipation can be neglected with considerable confidence in more complex analyses.

3. If all of the fluid between the plates is in the gas-like region above the critical point, conventional analogies between heat transfer and skin friction may do an adequate job of predicting heat transfer. If, however, any of the fluid between the plates is at the critical point, these analogies are not likely to produce success.

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